Nuclear Reaction Rates

The rate for a given reaction depends on the number density of the reactants, N_a and N_X, the rate these reactants encounter each other (*i.e.,* their velocity) and on the cross section for the reaction

$$\sigma = \frac{\text{number of reactions/nucleus } X \text{ /unit time}}{\text{number of incident particles/cm}^2\text{/unit time}}$$

Moreover, because of the electrostatic repulsion of the reactants (and because of atomic resonances) the cross section itself is a function of particle velocity. Thus

$$r_{aX} = N_a N_X \sigma(v) v$$

Of course, the particles in a star will have a distribution of velocities, hence in order to calculate the true reaction rate, we have to integrate over a velocity distribution, $\phi(v)$. In addition, if a and X are identical particles, the number of particle pairs will not be N_X^2, but only 1/2 this value. Thus the actual reaction rate will be

$$\begin{aligned} r_{aX} &= (1 + \delta_{aX})^{-1} \; N_a N_X \int_0^\infty \sigma(v)\, v\, \phi(v)\, dv \\ &= (1 + \delta_{aX})^{-1} \; \frac{\rho^2 N_A^2 X_a X_X}{A_a A_X} \; \langle \sigma v \rangle \end{aligned} \tag{11.1}$$

where δ_{aX} is the Kronecker delta, and N_A is Avogadro's number. For $\phi(v)$ we can substitute in the Maxwellian distribution, which, in the center of mass frame, is

$$\phi(v)\, dv = 4\pi v^2 \left(\frac{\mu}{2\pi kT}\right)^{3/2} \exp\left(-\frac{\mu v^2}{2kT}\right) dv$$

where μ is the reduced mass

$$\mu = \frac{M(a)M(X)}{M(a) + M(X)}$$

Thus, the reaction rate is

$$\begin{aligned}\langle \sigma v \rangle &= 4\pi \left(\frac{\mu}{2\pi kT} \right)^{3/2} \int_0^\infty v^3 \sigma(v) e^{-\mu v^2/2kT} dv \\ &= \left(\frac{8}{\pi \mu} \right)^{1/2} \left(\frac{1}{kT} \right)^{3/2} \int_0^\infty \sigma(E) E e^{-E/kT} dE \qquad (11.2)\end{aligned}$$

The value $\langle \sigma v \rangle$ itself depends on three factors:

- The probability of overcoming the coulomb barrier
- The probability of a quantum-mechanical interaction
- Whether the reaction occurs near a nuclear resonance

NON-RESONANT REACTIONS

Before any nuclear reaction can occur, a substantial potential barrier must be overcome. For a typical nuclear size of 1.2 fm= 10^{-13} cm, this barrier is equivalent to

$$V = \frac{Z_a Z_X e^2}{R} \sim kT; \qquad T \sim 1.4 \times 10^{10} K$$

This temperature is orders of magnitude higher than that found in stars. Thus, for nuclear reactions to proceed, particles must tunnel through the potential barrier. From quantum mechanics, the probability of doing this is

$$P \propto \exp \left\{ - \frac{2\pi Z_a Z_X e^2}{\hbar v} \right\} \propto \exp \left\{ - \frac{2(2\mu)^{1/2} \pi^2 Z_a Z_X e^2}{h E^{1/2}} \right\} \qquad (11.3)$$

Quantum mechanics also says that the cross section for two interacting particles is proportional to the de Broglie wavelength of the particles (since each particle sees the other as a smear over length $\lambda = \hbar/p$). Thus,

$$P \propto \pi\lambda^2 \propto \left(\frac{1}{p}\right)^2 \propto \frac{1}{E} \tag{11.4}$$

Thus, as long as there are no resonances,

$$\sigma = \frac{S(E)}{E} \exp\left\{-\frac{4\pi^2 Z_a Z_X e^2}{hv}\right\} \tag{11.5}$$

where the astrophysical cross-section $S(E)$ is a slowly varying function that contains factors which are intrinsic to the individual nucleus. With this definition, the nuclear reaction rate, as a function of temperature (11.2) becomes

$$\begin{aligned} \langle\sigma v\rangle &= \sqrt{\frac{8}{\pi\mu}}\left(\frac{1}{kT}\right)^{\frac{3}{2}} \int_0^\infty S(E)\exp\left\{-\frac{E}{kT} - \frac{2\sqrt{2\mu}\,\pi^2 Z_a Z_X e^2}{hE^{1/2}}\right\} dE \\ &= \sqrt{\frac{8}{\pi\mu}}\left(\frac{1}{kT}\right)^{\frac{3}{2}} \int_0^\infty S(E)\exp\left\{-\frac{E}{kT} - \frac{b}{E^{1/2}}\right\} dE \end{aligned} \tag{11.6}$$

where $b = 0.99 Z_1 Z_2 A^{1/2}$ MeV$^{1/2}$ and A is the reduced atomic weight in a.m.u.

We can evaluate (11.6) (or, at least, approximate its solution) using the method of steepest descent. First, note that the integrand is a sharply peaked exponential; by setting the derivative of the exponent to zero, it is easy to show that the "Gamow peak" occurs at

$$E_0 = \left(\frac{bkT}{2}\right)^{2/3} \tag{11.7}$$

Next, let's expand the argument of the exponential as a Taylor series about E_0, *i.e.*,

$$f(E) = -\frac{E}{kT} - \frac{b}{E^{1/2}} = f(E_0) + f^{'}(E_0)(E - E_0) + \frac{f^{''}(E_0)}{2!}(E - E_0)^2 +$$

If we substitute for b using (11.7), the first term is simply

$$-\tau = -\frac{E_0}{kT} - \frac{b}{E_0^{1/2}} = -\frac{3E_0}{kT}$$

The second term then disappears (by definition of $f^{'}(E_0)$). That leaves the third term:

$$\frac{f^{''}(E_0)}{2!}(E - E_0)^2 = -\frac{1}{2}\left(\frac{3b}{4E_0^{5/2}}\right)(E - E_0)^2$$

$$= -\frac{3b}{8}\left(\frac{2}{bkT}\right)^{\frac{5}{3}}(E - E_0)^2 = \frac{-3}{4E_0 kT}(E - E_0)^2$$

Now let $\Delta = 4\sqrt{E_0 kT/3}$. The reaction rate given by (11.6) then is

$$\langle \sigma v \rangle = \sqrt{\frac{8}{\pi\mu}}\left(\frac{1}{kT}\right)^{\frac{3}{2}} e^{-\tau} \int_0^{\infty} S(E) \exp\left[-\left(\frac{E - E_0}{\Delta/2}\right)^2\right] dE \tag{11.8}$$

Now let's look at the lower limit of the integral. For tempertures $T \lesssim 10^9$ K, one can show using (11.7) that $E_0 \gg \Delta$. This being the case, the lower end of the integral contributes very little to the result. We can therefore change the lower limit to negative infinity without much penalty, and make the integral analytic. Thus

$$\langle \sigma v \rangle = \left(\frac{8}{\pi\mu}\right)^{1/2}\left(\frac{1}{kT}\right)^{3/2} e^{-\tau} S(E_0) \pi^{1/2} \Delta \tag{11.9}$$

which, after a bit of simple math yields

$$\tau = 42.48 \left(\frac{Z_a^2 Z_X^2 A}{T_6} \right)^{1/3} = B\, T_6^{-1/3} \tag{11.10}$$

and

$$\langle \sigma v \rangle = K \left(A Z_a Z_X \right)^{-1} S(E_0)\, \tau^2 e^{-\tau} \tag{11.11}$$

where $K = 4.5 \times 10^{14}$ if $S(E_0)$ is measured in erg-cm^2, or 7.2×10^{-19} if it is given in the more common (at least for nuclear physics) units of keV-barns.

The previous analysis assumes that $S(E)$ is constant at $S(E_0)$ over the relevant range of the Gamow peak. We can improve upon this approximation by expanding $S(E)$ as a power series, and repeating the cross section calculation. If we substitute

$$S(E) = S(E_0) + \left(\frac{\partial S}{\partial E} \right)_{E_0} (E - E_0)$$

in (11.6) and do (quite a bit) of math, we find that, to first order in $1/\tau$, the result of (11.9) should be corrected by a factor of

$$G(\tau) = 1 + \frac{5}{2} \frac{E_0}{S(E_0)} \left(\frac{\partial S}{\partial E} \right)_{E_0} \frac{1}{\tau} = 1 + \frac{5}{6} \frac{kT}{S(E_0)} \left(\frac{\partial S}{\partial E} \right)_{E_0}$$

What this says, is that instead of using $S(E_0)$ in (11.9) and (11.11), we should define a variable S_0 and use that instead. This new variable is the weighted average of the astrophysical cross section over the peak, which is given by

$$\begin{aligned} S_0 = S(E_0) G(T) &= S(E_0) + \frac{5}{6} \left(\frac{\partial S}{\partial E} \right)_{E_0} kT \\ &\approx S(E_0 + \frac{5}{6} kT) \end{aligned} \tag{11.12}$$

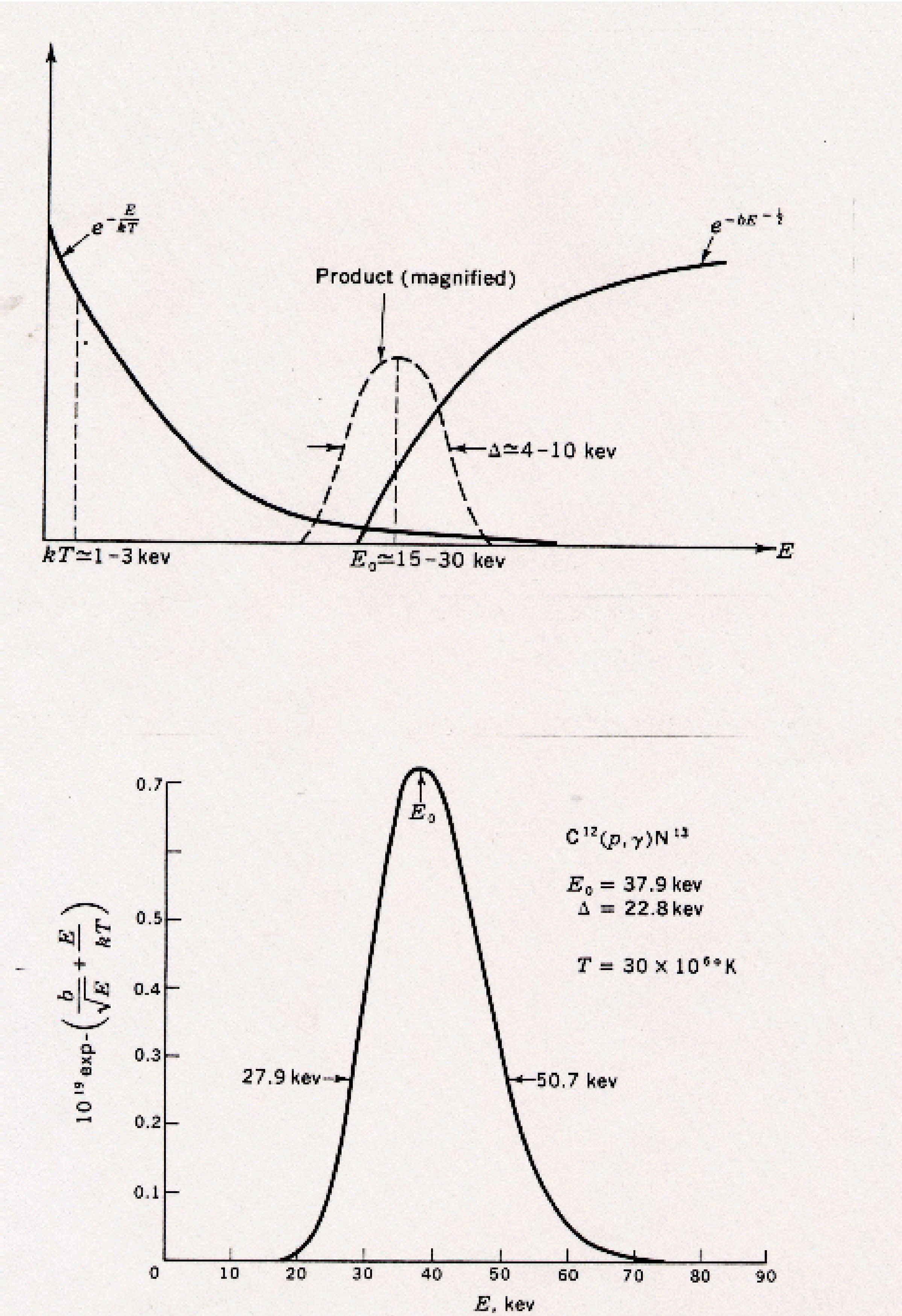
Note that, in this form of (11.9 - 11.11), the temperature dependence is entirely in the variable τ, which is proportional to $T^{-1/3}$. Formally,

$$\langle \sigma v \rangle = \frac{0.72 \times 10^{-18} a^2 S_0}{Z_a Z_X A} \; \frac{e^{-a T_6^{-1/3}}}{T_6^{2/3}} \; \mathrm{cm}^3 \; \mathrm{s}^{-1} \tag{11.13}$$

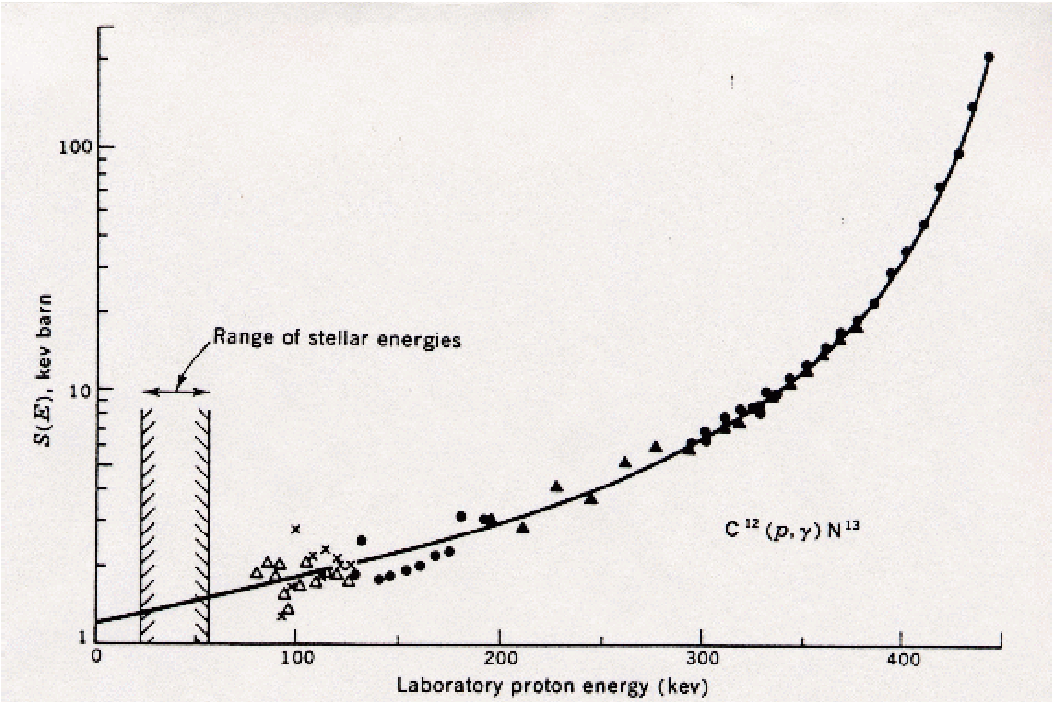
(where $a = 42.49\,(Z_a^2 Z_X^2 A)^{1/3}$ and S is in keV-barns), but we can see the temperature dependence better by taking the logarithmic derivative of $\langle \sigma v \rangle$ from (11.11), *i.e.*,

$$\begin{aligned} \nu = \frac{d \ln \langle \sigma v \rangle}{d \ln T} &= \left(\frac{d \ln \langle \sigma v \rangle}{d \ln \tau} \right) \left(\frac{d \ln \tau}{d \ln T} \right) \\ &= \frac{\tau - 2}{3} \\ &= \frac{14.16}{T_6^{1/3}} \left\{ Z_a^2 Z_X^2 A \right\}^{1/3} - \frac{2}{3} \end{aligned} \tag{11.14}$$

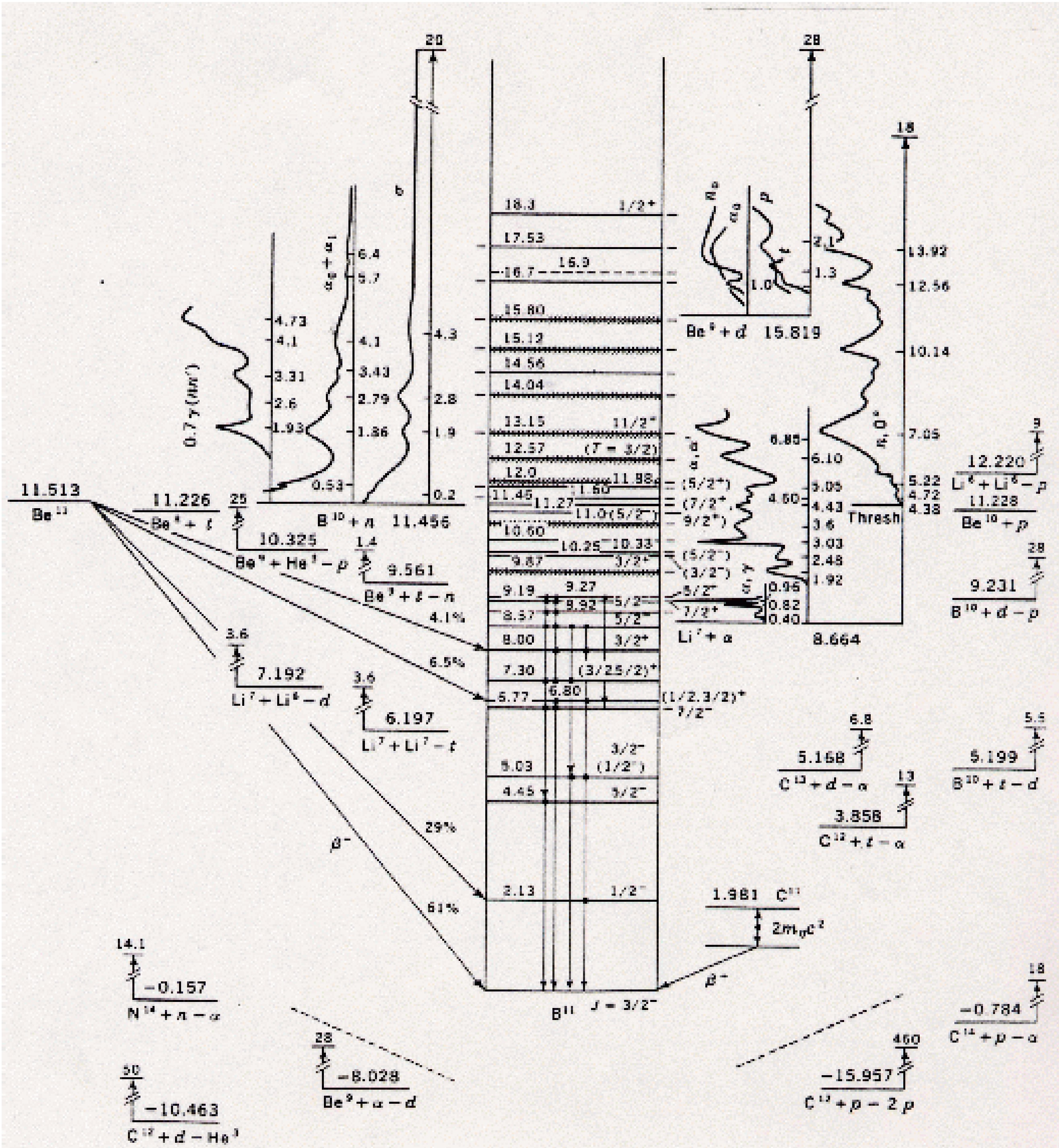
For typical central temperatures of stars (between 10^6 and 10^7 K), $\nu \gtrsim 5$. Nuclear reactions in stars are extremely sensitive to temperature!



The Gamow peak for non-resonant reactions



Experiments measuring λ for ${}^{12}\mathrm{C}(p,\gamma)^{13}\mathrm{N}$ vs. stellar conditions.



Energy level diagram for nucleus of ^{11}B.

RESONANT REACTIONS

The nucleus of an atom has energy levels similar to those of electrons. If the energy of an interacting particle coincides with the energy of one those levels, a resonance occurs, and the probability of interaction is boosted by several orders of magnitude. In this case, the off-resonance interactions (described above) become insignificant.

To compute the reaction rate for resonances, consider first that the reaction described in (10.4.1) is actually

$$a + X \longrightarrow Z^* \longrightarrow Y + b \tag{11.15}$$

When the reaction occurs, particle a and X form a compound nucleus Z, which is in an excited state. Just like an atom, this excited state can be described via a series of quantum numbers. Also, just like an atom, an excited nucleus will, after a very brief period of time, decay to a lower energy state. This lower energy state may be $Y + b$.

To estimate the strength of a nuclear resonance, let's first perform a semi-classical estimate of the maximum theoretical cross-section of nucleus X to particle a. First, let's say that the angular momentum quantum number of the Z^* state is ℓ. Classically, the cross section of X to a would be πb^2, where b is the impact parameter. In the realm of quantum mechanics, however, the impact parameter is given by the de Broglie wavelength of the particle pair, $\lambdabar = \hbar/p$. Since angular momentum is a quantized variable, this implies that the impact parameter is constrained so that $b = \ell\lambdabar$. Moreover, since angular momentum must be conserved, a collision which excites a nucleus to a state with angular momentum ℓ must taken place between the ℓ and $\ell + 1$ orbital. So

$$\sigma = \pi \left\{(\ell + 1)\,\lambdabar\right\}^2 - \pi(\ell\lambdabar)^2 = (2\ell + 1)\,\pi\lambdabar^2 \tag{11.16}$$

This argument (in addition to being non-rigorous) ignores particle spins, which interact with orbital angular momentum vectorially. In practice, the theoretical limit to the cross section is

$$\sigma = \pi \lambda\!\!\!^{-2}\, \omega = \pi \lambda\!\!\!^{-2} \; \frac{2J+1}{(2J_a+1)(2J_X+1)} \tag{11.17}$$

where J is the total angular momentum of the resonance state, and J_a and J_X are the total angular momenta of particles a and X.

Next, let's consider the stability of state Z^*. Through the uncertainty principle, there is a relation between a state's lifetime and its energy width, specifically, $\Gamma = \hbar/\tau$. Thus, the probability of a state decaying via method i (as opposed to all other methods) is

$$P_i = \frac{1/\tau_i}{\sum_j (1/\tau_j)} = \frac{\tau}{\tau_i} = \frac{\Gamma_i}{\Gamma} \tag{11.18}$$

Finally, consider the energy width of a given state. In exact analogy to electron orbitals, there is a finite energy width to a nuclear state; just like atoms, the width of the state is given by

$$f(E) = \frac{\Gamma^2}{(E - E_r)^2 + (\Gamma/2)^2} \tag{11.19}$$

where E_r is the mean energy of the state. (Again, this is due to the uncertainty principle. The total energy of the state is indeterminate; we only have a probability estimate.)

We can now estimate the nuclear resonance cross section. This cross section involve three terms. The first is simply the energy width of the state. The probability of an interaction depends on the exact energies of the incoming particles: the closer to the mean energy of the resonance, the higher the probability of interaction, *i.e.*, $\sigma \propto f(E)$. Next, one must consider the probability of a actually decaying into state Z^* as opposed to all other states. This probability is simply Γ_a/Γ. Finally, the probability of state Z^* doing anything *except decaying back into particle* a (and therefore not producing any reaction) is $(\Gamma - \Gamma_a)/\Gamma$. Therefore

$$\sigma = \pi \lambdabar^2 \omega \, \frac{\Gamma^2}{(E - E_r)^2 + (\Gamma/2)^2} \left(\frac{\Gamma_a}{\Gamma}\right) \left(\frac{\Gamma - \Gamma_a}{\Gamma}\right)$$

or

$$\sigma = \pi \lambdabar^2 \omega \frac{\Gamma_a(\Gamma - \Gamma_a)}{(E - E_r)^2 + (\Gamma/2)^2} \tag{11.20}$$

If we substitute for $\lambdabar = \hbar/p$ using $p = \sqrt{2\mu E}$, and if there is only one possible decay option (*i.e.*, $\Gamma_b = \Gamma - \Gamma_a$), then

$$\sigma(E) = \pi\omega \left(\frac{\hbar^2}{2\mu E}\right) \frac{\Gamma_a \Gamma_b}{(E - E_r)^2 + (\Gamma/2)^2} \tag{11.21}$$

(This is the Breit-Wigner single-level formula.)

To estimate the rate of a resonance reaction in stars, $\sigma(E)$ can be substituted into the integral of (11.2). Over the width of a typical resonance ($\sim$ 1 eV), the Maxwellian distribution barely changes, hence we can adopt its value at the resonance, *i.e.*,

$$\begin{aligned}
\langle\sigma v\rangle &= \sqrt{\frac{8}{\pi\mu}}\left(\frac{1}{kT}\right)^{\frac{3}{2}}\int_0^\infty \sigma(E)Ee^{-E/kT}dE \\
&= \sqrt{\frac{8}{\pi\mu}}\left(\frac{1}{kT}\right)^{\frac{3}{2}}\int_0^\infty \left(\frac{\pi\omega\hbar^2}{2\mu E}\right)\frac{\Gamma_a\Gamma_b Ee^{-E/kT}}{(E-E_r)^2+(\Gamma/2)^2}dE \\
&= \sqrt{2\pi}\left(\frac{1}{\mu kT}\right)^{\frac{3}{2}}\omega\hbar^2\Gamma_a\Gamma_b e^{-E_r/kT}\int_0^\infty \frac{dE}{(E-E_r)^2+(\Gamma/2)^2}
\end{aligned}$$

For simplicity, we can again take the lower limit on the integral to negative infinity, *i.e.*,

$$\int_{-\infty}^{\infty}\frac{dE}{(E-E_r)^2+(\Gamma/2)^2} = \frac{1}{\Gamma/2}\tan^{-1}\left(\frac{E}{\Gamma/2}\right)\Bigg]_{-\infty}^{\infty} = \frac{2\pi}{\Gamma}$$

This gives us

$$\begin{aligned}
\langle\sigma v\rangle &= \hbar^2\left(\frac{2\pi}{\mu kT}\right)^{3/2}\left\{\omega\frac{\Gamma_a\Gamma_X}{\Gamma}\right\}e^{-E_r/kT} \\
&= 8.08\times 10^{-9}\,(AT_6)^{-3/2}\ (\omega\gamma)_r\ e^{-11605E_r/T_6} \qquad (11.22)
\end{aligned}$$

where E_r and $(\omega\gamma)_r = \omega\frac{\Gamma_a\Gamma_x}{\Gamma}$ are given in MeV, and the reduced mass A is given in atomic mass units. Again, there is an extremely steep temperature dependence, with

$$\nu = \frac{d\ln\langle\sigma v\rangle}{d\ln T} = \frac{11605E_r}{T_6} - \frac{3}{2} \qquad (11.23)$$

For typical energies of ~ 0.1 MeV, and temperatures of $T_6 \sim 50$, $\nu \sim 20$.

A few points

- In astrophysical situations, reactions sometimes change from one regime to another, depending on the temperature of the star, and the energy of the resonance.

- Sometimes, it is very difficult to find all the resonances of a nucleus in the lab, and if a narrow resonance is missed, the quoted reaction rate will be substantially off.

- Heavy elements have many more resonances than light elements; by the time you get into the third row of the periodic table, all the reactions proceed through (overlapping) resonances.

- If particle a is a neutron, then there is no Coulomb barrier to overcome, and the cross section for capture is nearly independent of energy. Neutron reactions are only important for heavier elements ($A \gtrsim 60$) and are not energetically important. (Their rate is primarily determined by the number of free neutrons in the star.)

- Some reactions (including the initial reaction in the proton-proton chain) involved the weak nuclear force. The temperature dependence of these reactions may be different from either of the forms presented above. (They depend on overlapping wave functions and the strength of the individual reaction.)

Summary for Reactions

When electron shielding is included, the nuclear reaction rate equation becomes

$$r_{aX} = (1+\delta_{aX})^{-1}\ \frac{\rho^2 X_a X_X N_A^2}{A_a A_X}\ f\ \langle\sigma v\rangle \tag{12.2.1}$$

or, in terms of number density

$$r_{aX} = (1+\delta_{aX})^{-1} N_a N_X f \langle\sigma v\rangle \tag{12.2.2}$$

For non-resonant reactions, $\langle\sigma v\rangle$ can be computed directly from $S(E_0)$, $\left(\frac{\partial S}{\partial E}\right)_{E_0}$, Z_a, Z_X, A_a, A_X, and T, while for resonant reactions, $\langle\sigma v\rangle$ is derived from $(\omega\gamma)$, E_r, and T.

Now let's simplify the terminology by first assigning

$$\lambda_{aX} = \langle\sigma v\rangle \tag{12.2.3}$$

and then defining the lifetime of species X against reactions with particle a to be

$$\frac{dN_X}{dt} = -r_{aX} = -\frac{N_X}{\tau_a} \tag{12.2.4}$$

The form of this equation is now identical with that associated with radioactivity or other decay phenomena. (Unlike radioactive decay, of course, τ_a depends on the external environment.) From the equation above,

$$\tau_a(X)\ =\ (1+\delta_{aX})^{-1}\frac{N_X}{r_{aX}}\ =\ (\lambda_{aX} f N_a)^{-1} \tag{12.2.5}$$

Note that the Kronecker delta disappears from the final lifetime calculation. According to (12.2.1), the reaction rate for identical

particles has a factor of two in the denominator, but since each reaction destroys two particles, this factor cancels out. Note also, that by (12.2.4) and (12.2.5), the total lifetime of particle X from all reactions is

$$\frac{1}{\tau} = \sum_i \frac{1}{\tau_i} \tag{12.2.6}$$